Fractionally Quantized Cooper Pair Stair Case in Superconducting Quantum Dots

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Abstract

We study clean superconducting quantum dots (SQD) and also site dependent Josephson couplings, on site charging energies and the intersite interactions in presence of gate voltage. We predict the existence of different fractionally quantized Cooper pair stair case with many interesting physical properties. The appearance of stair case is not only due to the Coulomb blocked phenomena but also for the site dependent Josephson couplings. We also explain physically the absence of other fractionally quantized Cooper pair stair case. The physics of fractionally quantized Cooper pair stair case has close resemblance with the fractionally quantized magnetization plateau of a spin chain system under a magnetic field.

Keyword's: Superconducting quantum dots, Fractionally quantized plateau, Abelian bosonization and Renormalization group study

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1. INTRODUCTION

In last few decades mesoscopic physics of nanoscale superconducting systems are revealing many interesting properties. There has been intense research aimed at developing the superconducting flux based digital electronics and computers [1, 2]. The single Cooper pair box (SCB) and single Cooper pair transistors have developed experimentally and used to demonstrate the quantization of Cooper pair on a small superconducting island [3], which is the foundation of charge qubit [4].

Here we mention briefly the mechanism of Cooper pair quantization for SCB (= Josephson junction circuit consisting of a small superconducting island connected via a Josephson tunnel junction to a large superconducting island). The SCB was first experimentally realized by Lafarge *et al*. [5] observing the Coulomb stair case with the step of $2e$. The evidence of Coulomb stair case has also predicted in Ref.[6]. Realization of the first charge qubit by manipulation of SCB and the observation of Rabi oscillation was done by Nakamura *et al* [7]. Our operating point is in the charge regime, i.e. $E_c >> E_J$. We consider a finite numbers of Cooper pairs (n) in the mesoscopic island. The eigen state equation of SCB is

$$E_c(\hat{n} - n_g)^2 |n >= E_n |n > \tag{1}$$

$n = 0, 1, 2$ corresponds to the charge state with energy spectrum $E_n = E_c(\hat{n} - n_g)^2$. n_g is the gate voltage induced charge in SQD. From Eq. 1, it is clear to us for a specific value of gate voltage (i.e $n_g = 1/2$) the charge state $|0>$ and $|1>$ become degenerate. Switching on a small Josephson coupling will then lift the degeneracy and forming a two level system. So the system reduced to the qubit system.

$$H_{SCB} = -1/2(B_z \sigma_z + B_x \sigma_x), \tag{2}$$

where $B_z = E_c(1 - 2n_g)$ and $B_x = E_J$. The qubit level energies are given by the equation $E_{1,2} = \pm (1/2)\sqrt{{E_c}^2 (1 - 2n_g)^2 + {E_J}^2}$. At the charge degeneracy point, $n_g = 1/2$, at this point only the off diagonal part contribute and the energy levels are separated by E_J and the qubit eigen states $|E_1 >= |0> - |1>$ and $|E_2 >= |0> + |1>$. For these states, the average charge on the island is zero, while it changes to $\pm 2e$ from the degeneracy point, where the qubit eigen state approach the pure charge state.

It is clear to us from the analyses of previous studies that the Cooper pair stair case (charge quantization state of SCB) only occurs due to presence of Coulomb blocked phenomena. Here we consider a model systems consists of array of clean superconducting quantum dots (SQD) and also a SQD systems with site dependent Josephson couplings, on-site charging energies and the intersite interactions. We will study different fractionally quantized Cooper pair stair case. It is clear to us from Eq. 1 that the basic physics of SCB can be understood by the spin chain model Hamiltonian under a magnetic field. We will see from the analysis of our model that the factionally quantized Cooper pair stair case is nothing but the fractionally quantized magnetization plateau state of spin chain system.

2. COOPER PAIR STAIR CASE AND PHYSICAL ANALYSIS OF COOPER OF CLEAN SUPERCONDUCTING QUANTUM DOTS

The model Hamiltonian of SQD system has different parts,

$$H = H_{J1} + H_{J2} + H_{EC0} + H_{EC1} + H_{EC2}. \tag{3}$$

We recast the different parts of the Hamiltonian in Quantum Phase model.
$H_{J1} = - E_{J1} \sum_i cos(\phi_{i+1} - \phi_i)$, $H_{J2} = - E_{J2} \sum_i cos(\phi_{i+2} - \phi_i)$.
Hamiltonians H_{J1} and H_{J2} are Josephson energy Hamiltonians respectively for nearest neighbor (NN) and next-nearest-neighbor (NNN) Josephson tunneling between the SQD. $H_{EC0} = \frac{E_{C0}}{2} \sum_i (-i\frac{\partial}{\partial\phi_i} - \frac{N}{2})^2$, $H_{EC1} = E_{Z1} \sum_i n_i \; n_{i+1}$, $H_{EC2} = E_{Z2} \sum_i n_i \; n_{i+2}$. H_{EC0}, H_{EC1}, and H_{EC2} are respectively the Hamiltonians for on-site, NN and NNN charging energies of the SQD. In the phase representation, $(-i\frac{\partial}{\partial\phi_i})$ is the operator representing the number of Cooper pairs at the ith dot, and thus it takes only the integer values (n_i). Hamiltonian H_{EC0} accounts for the influence of gate voltage $(eN \sim V_g)$. eN is the average dot charge induced by the gate voltage. When the ratio $\frac{E_{J1}}{E_{C0}} \rightarrow 0$, the SQD array is in the insulating state having a gap of the width $\sim E_{C0}$, since it costs an energy $\sim E_{C0}$ to change the number of pairs at any dot. The exception are the discrete points at $N = 2n+1$, where a dot with charge $2ne$ and $2(n+1)e$ has the same energy because gate charge compensates the charges of extra Cooper pair in the dot. On this degeneracy point, a small amount of Josephson coupling leads the system to the superconducting state. We are interested in analyzing the phases explicitly near this degeneracy point.

Now we want to recast our basic Hamiltonians in the spin language. During this process we follow Ref. ([8]). $H_{J1} \; = \; -2 \; E_{J1} \sum_i (S_i{}^{\dagger} S_{i+1}{}^{-} + h.c)$, $H_{J2} \; = \; -2 \; E_{J2} \sum_i (S_i{}^{\dagger} S_{i+2}{}^{-} + h.c)$ $H_{EC0} \; = \; \frac{E_{C0}}{2} \sum_i (2S_i{}^Z - h)^2$.

$$H_{EC1} \; = \; 4E_{Z1} \sum_i S_i{}^Z \; S_{i+1}{}^Z,$$

$$H_{EC2} \; = \; 4E_{Z2} \sum_i S_i{}^Z \; S_{i+2}{}^Z.$$

Here $h = \frac{(N-2n-1)}{2}$ allows tuning of the system to a degeneracy point by means of gate voltage. The phase diagram is periodic in N with period 2. Here we only consider a single slab of the phase diagram, $0 \leq N \leq 2$. One can express spin chain systems to spinless fermions systems through the application of Jordan-Wigner transformation. In Jordan-Wigner transformation the relations between the spin and the electron creation and annihilation operators are [9]

$$S_i^z = \psi_i^{\dagger}\psi_i - 1/2 \; , S_i^- = (-1)^i \psi_i \; \exp[i\pi \sum_{j=-\infty}^{i-1} n_j] \; , S_i^+ = (S_i^-)^{\dagger}, \tag{4}$$

where $n_j = \psi_j^{\dagger}\psi_j$ is the fermion number at site j. Therefore,

$$H_{J1} = -2E_{J1} \sum_i \; (\psi_{i+1}^{\dagger}\psi_i + \psi_i^{\dagger}\psi_{i+1}), \tag{5}$$

$$H_{J2} = -2E_{J2} \sum_i \; (\psi_{i+2}^{\dagger}\psi_i + \text{h.c.})(\psi_{i+1}^{\dagger}\psi_{i+1} - 1/2), \tag{6}$$

$$H_{EC0} \; = \; -2hE_{C0} \sum_i \; (\psi_i^{\dagger}\psi_i - 1/2), \tag{7}$$

$$H_{EC1} \; = \; 4E_{Z1} \sum_i (\psi_i^{\dagger}\psi_i - 1/2)(\psi_{i+1}^{\dagger}\psi_{i+1} - 1/2), \tag{8}$$

$$H_{EC2} \; = \; 4E_{Z2} \sum_i \; (\psi_i^{\dagger}\psi_i - 1/2)(\psi_{i+2}^{\dagger}\psi_{i+2} - 1/2). \tag{9}$$

In order to study the continuum field theory of these Hamiltonians, we recast the spinless fermions operators in terms of field operators by this relation [9].

$$\psi(x) \; = \; [e^{ik_F x} \; \psi_R(x) \; + \; e^{-ik_F x} \; \psi_L(x)] \tag{10}$$

where $\psi_R(x)$ and $\psi_L(x)$ describe the second-quantized fields of right- and the left-moving fermions respectively. k_F is Fermi wave vector. It reveals from Eq. 7 that the applied external gate voltage on the dot systems appears as a magnetic field in spin chain. So the different values of applied gate voltage promote the system in the different state of magnetization. In our system k_F will depend on the applied gate voltage. The Fermi wave vector and the magnetization state (m) are related by the relation $k_F = \frac{\pi}{2}(1 - 2m)$. We would like to study the effect of gate voltage in our study so we keep k_F as arbitrary. In our study, we will mainly focus on the magnetization state with $m = 0, 1/4, 1/2$. We would like to express the fermionic fields in terms of bosonic field by the relation

$$\psi_r(x) \; = \; \frac{U_r}{\sqrt{2\pi\alpha}} \; e^{-i \; (r\phi(x) \; - \; \theta(x))}, \tag{11}$$

r is denoting the chirality of the fermionic fields, right (1) or left movers (-1). The operators U_r commutes with the bosonic field. U_r of different species commute and U_r of the same species anti-commute. ϕ field corresponds to the quantum fluctuations (bosonic) of spin and θ is the dual field of ϕ. They are related by the relations $\phi_R \; = \; \theta \; - \; \phi$ and $\phi_L \; = \; \theta \; + \; \phi$. The model Hamiltonian after continuum field theoretical studies for arbitrary values of k_F is

$$\begin{aligned} H \; &= \; H_0 + \; \frac{4E_{Z1}}{(2\pi\alpha)^2} \int \; : cos(4\sqrt{K}\phi(x) \; - (G - 4k_F)x - 2k_F a) : \; dx \\ &+ \; \frac{4E_{Z2}}{(2\pi\alpha)^2} \int \; : cos(4\sqrt{K}\phi(x) \; + (G - 4k_F)x - 4k_F a) : \; dx \end{aligned} \tag{12}$$

$$\begin{aligned} H_0 \; &= \; (\frac{v}{2\pi} + \frac{8E_{C0}}{\pi^2} - 4E_{J2} - \frac{2{E_{J1}}^2}{E_{C0}}) \; \int dx \; [: (\partial_x\theta)^2 : + : (\partial_x\phi)^2 : \;] \\ &+ (16E_{C0} - 8E_{J2} - 4\frac{{E_{J1}}^2}{E_{C0}}) \int dx \;\; : (\partial_x\theta - \partial_x\phi)(\partial_x\theta + \partial_x\phi) : \end{aligned} \tag{13}$$

H_0 is the non-interacting part of the model Hamiltonian, v is the velocity of low energy excitations liquid parameter and the other is K $(= \sqrt{[\frac{E_{J1} \; - \; \frac{32}{\pi}E_{J1}E_{Z2}}{E_{J1} \; + \; \frac{2}{\pi}(4E_{Z1} \; - \; \frac{3{E_{J1}}^2}{4E_{C0}})}]})$.

2.1 CALCULATIONS AND RESULTS FOR COOPER PAIR STAIR CASE FOR SINGLE PAIR IN ALTERNATE SITES ($m = 0$ MAGNETIZATION PLATEAU)

In this sub-section we discuss the Cooper pair stair case for a single pair in alternate sites. We have already mentioned that the SQD array system is nothing but the spin-1/2 chain and

stair case of the SQD is nothing but the magnetization plateau. We will see that at $m = 0$ ($k_F = \pi/2$) there is an evidence of magnetization plateau. We would like to mention basic criteria for the appearance of magnetization plateau before we start our full swing discussion on stair case physics. The basic criteria for the appearance of plateau is the following: It is well known to us that the elementary excitation of an one dimensional spin system is gapless for half integer spin chain and gapped for integer spin chain. In the presence of magnetic field, it is possible for an integer spin chain to be gapless with partial magnetization and a half-odd-integer spin chain to show a gap above the ground state for appropriate values of the magnetic field. It has been shown by different groups [10, 11, 12, 13, 14] that the magnetization of some systems can exhibit plateaus at certain nonzero values for some finite ranges of the magnetic field. The basic criteria for the appearance of magnetization plateau can be understood from the extension of Lieb-Schultz-Mattis theorem under a magnetic field. This implies that translationally invariant spin chains in an applied field can be gapful without breaking translation symmetry under the condition $S - m =$ integer, where S and m are the spin and the magnetization state of the chain. In this gapful phase magnetization plateau occurs for quantized values of m. Fractional quantization can also occur, if accompanied by spontaneous breaking of translational symmetry. Fractional quantization can be understood from the $S - m =$ non integer condition. In this situation system is either in the gapless low lying states or the degenerate ground state with spontaneous translational symmetry breaking in the thermodynamic limit. These conditions for the appearance of plateau are the necessary but not the sufficient condition. The nature and the occurrence of the plateau are also dependent on the nature of interaction present in the system [15, 16].

Our model Hamiltonian for $m = 0$ magnetization plateau, in terms of bosonic fields, are

$$H = H_0 \; - \; \frac{4(E_{Z1} - E_{Z2})}{(2\pi\alpha)^2} \int cos(4\sqrt{K}\phi(x) \;) \; dx \tag{14}$$

In the spin chain, system is in the Neel-Ising state when XXZ anisotropy is greater than 1 ($4E_{Z1} > 1$). This state is doubly degenerate state and $S - m = 1/2$, so this phase of the system satisfy the all criteria of fractionally quantized magnetization plateau. This is the relevant physics when the NNN exchange interaction is less than a critical value. When NNN exchange interaction exceed some critical value, ground state of the system is dimerized and doubly degenerate. The dimerized ground state is the product of spin singlet of adjacent sites [17, 18]. In both ordering the lattice translational symmetry is breaking.

In our clean SQD system, we have predicted the CDW wave state with lattice translational symmetry breaking is alike to Neel-Ising state of one dimensional spin systems. This CDW state is the one-dimensional counter part of the two-dimensional checker board phase. One can find the dimer order density wave for the lower values of Cooper pair density, when one consider further long range interaction between the SQD. To get the dimer-order density in the presence of NNN interactions between dots, we shall have to consider the charge degeneracy point at higher values of half-integer Cooper pair density like $n = 3/2$. This Cooper pair stair case is originating due to the intersite Coulomb repulsion, so it is related with the Coulomb blocked phenomena of the system.

2.2 CALCULATIONS AND RESULTS FOR COOPER PAIR STAIR CASE FOR SINGLE PAIR IN EVERY TWO SITES ($m = 1/4$ MAGNETIZATION PLATEAU)

In this sub-section we discuss the occurrence of finite magnetization plateau ($m = 1/4$). This plateau state correspondence to the Cooper pair stair case for single Cooper pair in every two sites. For weak dimerization, i.e., when there is no reduction of unit cell size, reciprocal lattice vector is 2π then the Hamiltonian of the system reduced to Eq. 15. None of the sine-Gordon coupling terms will provide the significant contributions for this plateau state, due to the oscillatory nature of the integrand. Effective Hamiltonian of the system for this state is the following:

$$\begin{aligned} H \;=\; H_0 \;-\; & \frac{4E_{Z1}}{(2\pi\alpha)^2}\int (-1)^x \; cos(4\sqrt{K}\phi(x)\;)\; dx \\ + \; & \frac{4E_{Z2}}{(2\pi\alpha)^2}\int (-1)^x \; cos(4\sqrt{K}\phi(x)\;)\; dx \end{aligned} \tag{15}$$

For strong dimerization state, reciprocal lattice vector G reduced to π due to the reduction of the unit cell. The model Hamiltonian of the system reduced to

$$H = H_0 \;-\; \frac{4E_{Z12}}{(2\pi\alpha)^2}\int cos(4\sqrt{K}\phi(x) \;+\; \beta)\; dx, \tag{16}$$

where $E_{Z12} = \sqrt{{E_{Z1}}^2 + {E_{Z2}}^2}$ and $\beta = tan^{-1}\frac{E_{z2}}{E_{Z1}}$. To study the renormalization group flow diagram, we shall have to construct the renormalization group (RG) equations. The RG equations of H_2 is

$$\frac{dK(l)}{dl} \;=\; -{E_{Z12}}^2 K(l)^2$$

$$\frac{dE_{Z12}}{dl} = (2 - 4K(l)\ )\ E_{Z12} \tag{17}$$

In Fig. 1, we present the renormalization group flow diagram. RG flow lines, flowing off to $E_{Z12} \rightarrow\ \infty$ are an Ising-Neel state, an anti-ferromagnetic ordering is setup in the z direction, i.e., the system is in the CDW wave state with only one boson in every two sites. This is the, B1 phase in the RG flow diagram. When the RG flow lines, flowing off to $E_{Z12} \rightarrow\ -\infty$ are the dimer order density waves. In this state two Cooper pair of adjacent sites are bound with each other. This is the insulating state of the system. In spin chain, dimerized ground state is the analogous state of this phase. This phase is depicted as a B2 phase in the RG flow diagram. Phase D is the repulsive Luttinger liquid phase, i.e., the gap-less phase due to the irrelevancy of sine-Gordon coupling terms. Phase E is the superconducting phase of the system. This Cooper pair stair case is originating due to the intersite Coulomb repulsion, so it is related with the Coulomb blocked phenomena of the system.

2.3 CALCULATIONS AND RESULTS FOR COOPER PAIR STAIR CASE AT EMPTY BAND LIMIT ($m = \frac{1}{2}$ MAGNETIZATION PLATEAU):

Now we discuss the saturation plateau at $m = \frac{1}{2}$ ($k_F = 0$). $K_F = 0$ implies that the band is empty and the dispersion is not linear, so the validity of the continuum field theory is questionable. Values of the two Luttinger liquid parameters, v_0 and K, are 0 and 1 respectively. It also implies that none of the sine-Gordon coupling terms become relevant in this parameter space. Saturation plateaus are only appearing due to very high values of magnetic field. In this plateau, system is in ferromagnetic ground state and restore the lattice translational symmetry. We think, this is the classical phase of the system.

Absence of other fractionally quantized magnetization plateaus: Here we present the possible explanation for the absence of other fractionally quantized magnetization plateaus (like $\frac{1}{3}, \frac{1}{5}$ etc): A careful examination of Eq. 12 reveals that to get a non oscillatory contribution from Hamiltonian one has to be satisfied $4k_F = G$ condition but this condition is not fulfilled for these plateaus. The integrand of this sine-Gordon coupling terms contain an oscillatory factor that leads to a vanishing contribution. The other criteria is that non vanishing sine-Gordon coupling term should be relevant.

3. MODEL HAMILTONIAN AND PHYSICAL ANALYSIS OF COOPER PAIR STAIR CASE OF SITE DEPENDENT COUPLINGS, SUPERCONDUCTING QUANTUM DOTS

At first we write down the model Hamiltonian of SQD system with site dependent Josephson couplings, on-site charging energies and intersite interactions in presence of gate voltage.

$$H \;=\; H_{J1} \;+\; H_{J2} \;+\; H_{EC0} \;+\; H_{EC1} \;+\; H_{EC2}. \tag{18}$$

We recast the different parts of the Hamiltonian in Quantum Phase model.

$$H_{J1} \;=\; -\,E_{J1} \textstyle\sum_i (1-(-1)^i \delta_1) cos(\phi_{i+1}-\phi_i),\; H_{J2} \;=\; -\,E_{J2} \textstyle\sum_i cos(\phi_{i+2}-\phi_i).$$

Here NN Josephson couplings are different on alternate sites. $E_{J1}(1+\delta_1)$ and $E_{J1}(1-\delta_1)$ are the Josephson coupling strength for odd and even site respectively.

$$H_{EC0} \;=\; \tfrac{E_{C0}}{2} \textstyle\sum_i (1-(-1)^i\delta_2)(-i\tfrac{\partial}{\partial\phi_i} - \tfrac{N}{2})^2,$$

$E_{C0}(1{+}\delta_2)$ and $E_{C0}(1{-}\delta_2)$ are the on site charging energies for odd and even site respectively.

$$H_{EC1} \;=\; E_{Z1} \sum_i (1-(-1)^i \delta_3) n_i \; n_{i+1},$$

$E_{Z1}(1+\delta_3)$ and $E_{Z1}(1-\delta_3)$ are the inter-site charging energies for odd and even site respectively. All δ's are the deviational parameter from the clean superconducting quantum dots.

$$H_{EC2} \;=\; E_{Z2} \sum_i n_i \; n_{i+2},$$

H_{EC0}, H_{EC1}, and H_{EC2} are respectively the Hamiltonians for on-site, NN and NNN charging energies. Following the prescription of previous sections, one can write the continuum model Hamiltonian.

$$\begin{aligned} H \;=\; & H_0 - 2E_{J1} \int dx : sin(2\sqrt{K}\phi(x) - (2k_F - \pi)x) : \\ & + \frac{4E_{Z1}}{(2\pi\alpha)^2} \int : cos(4\sqrt{K}\phi(x) \;-(G-4k_F)x - 2k_F a) : \; dx \\ & + \frac{4E_{Z1}\delta_3}{(2\pi\alpha)^2} \int (-1)^x : cos(4\sqrt{K}\phi(x) \;-(G-4k_F)x - 2k_F a) : \; dx \\ & + \frac{2hE_{C0}}{\pi\alpha} \int (-1)^x : sin(2\sqrt{K}\phi(x) + 2k_F x) : \; dx \\ & + \frac{4E_{Z2}}{(2\pi\alpha)^2} \int : cos(4\sqrt{K}\phi(x) \;+(G-4k_F)x - 4k_F a) : \; dx \end{aligned} \tag{19}$$

3.1 CALCULATION AND RESULTS FOR COOPER PAIR STAIR CASE OF A SITE DEPENDENT SQD ARRAY FOR SINGLE COOPER PAIR IN ALTERNATE SITES ($m = 0$ MAGNETIZATION PLATEAU)

In this sub section, we find the evidence of Cooper pair stair case for single pair in alternate site for site dependent SQD system. The effective Hamiltonian reduce to

$$\begin{aligned} H \;=\; & H_0 + 2E_{J1}\delta_1 \int \, dx : cos(2\sqrt(K)\phi(x)) : +2\frac{hE_{C0}}{\pi\alpha}\delta_2 \int \, dx : cos(2\sqrt(K)\phi(x)) : \\ & -\frac{4E_{Z12}}{(2\pi\alpha)^2} \int \, dx : cos(4\sqrt{K}\phi(x)) : \end{aligned} \tag{20}$$

At around the charge degeneracy point, the system is in the mixed state (M), i.e., the simultaneous presence of dimer density wave and staggered phase. Our model Hamiltonian consists of three sine-Gordon couplings. The first one arises due to site dependent variations of NN Josephson couplings, it yields the dimerized phase of the system. The anamolous scaling dimension of this term is $2K$. This phase is spontaneous, i.e., infinitesimal variation of NN Josephson coupling is sufficient to produce this state. We have predicted in the previous sections of this work that the stair case is appearing due to the Coulomb blocked phenomena. Here we observe that the stair case may also occur for the Josephson tunneling. This prediction was absent in the previous studies. This observation of dimerized phase is in contrast with the clean superconducting quantum dots array, where it appears when NNN Josephson coupling exceed some critical value. The second sine-Gordon coupling term arises due to the site dependent applied gate voltage. The anamolous scaling dimension of this term is $2K$. It yields the staggered phase of the system. The system is in the mixed phase (M) when both of the couplings are in equal magnitude otherwise the system is in any one of the state of this mixed phase, depending on the strength of couplings. We have ignored the 3rd sine-Gordon coupling term of the Hamiltonian, because the anamolous scaling dimension is much larger than the other two.

3.2 CALCULATION AND RESULTS FOR COOPER PAIR STAIR CASE OF A SITE DEPENDENT SQD ARRAY FOR SINGLE COOPER PAIR IN EVERY TWO SITES ($m = 1/4$ MAGNETIZATION PLATEAU) AND EMPTY BAND LIMIT ($m = 1/2$ MAGNETIZATION PLATEAU)

In this sub section, we find the evidence of Cooper pair stair case for single pair in every two sites and also for empty band limit of site dependent SQD system.

$m = 1/4$: in the weak dimerization limit, Hamiltonian reduced to

$$H = H_0 - \frac{4E_{Z1}\ \delta_3}{(2\pi\alpha)^2} \int\ dx : sin(4\sqrt{K}\phi(x)) : \tag{21}$$

The sine-Gordon coupling term became relevant when $K < 1/2$. If we analyze the expression for Luttinger liquid parameter then we get the following quadratic equation of the system parameters to achieve this stair case phase for weak dimerization limit.

$$E_{J1}{}^2 - E_{J1}(4 - \frac{512}{3}E_{Z1}) - 4E_{Z1} \leq 0$$

This phase is in contrast with the stair case phase of the homogeneous SQD. In weak dimerization, there is no evidence of stair case phase for homogeneous SQD system. This phase of the system satisfies the all criteria of fractional quantization. Now we discuss for stronger strength of dimerization, i.e., when the dimerization strength exceed some critical value, at this point reciprocal lattice vector G reduced from 2π to π. The Hamiltonian of the system reduced to

$$H = H_0 - \frac{4E_{Z12}}{(2\pi\alpha)^2} \int\ dx : cos(4\sqrt{K}\phi(x) + \beta) : \tag{22}$$

Expression for E_{Z12} and β has given in section 2.2. This equation is identical to Eq. 16. So the physics of the system is the same as Fig. 1. Hence the physical behavior of the system in strong dimerization limit is the same for homogeneous and site dependent SQD.

$m = 1/2$ and other fractionally quantized plateaus: In these phases non of the sine-Gordon coupling terms are relevant. So there is no plateau phases for this states of the system. Hence the physical behavior of the system in strong dimerization limit is the same for homogeneous and site dependent SQD.

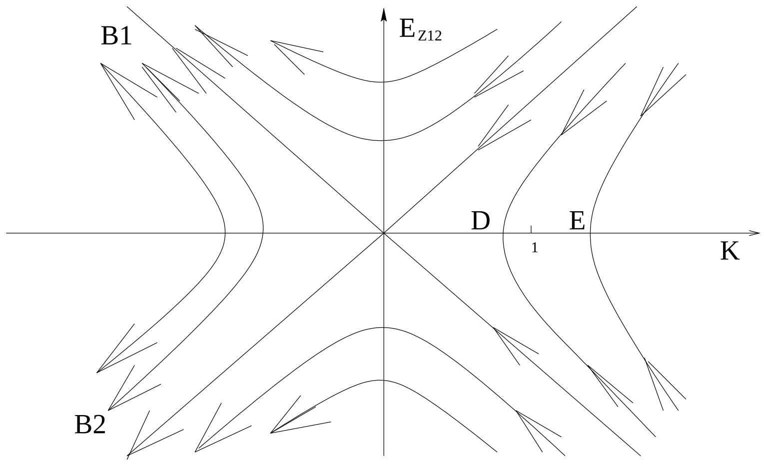

FIG. 1: Renormalization group flow diagram of the SQD array. We have depicted the different phases of the model Hamiltonians by: B1. Charge-density wave (CDW), B2. Dimer-order density wave; D. Second kind of Repulsive Luttinger liquid; E. Superconducting phase.

4. CONCLUSIONS

We have predicted the Cooper pair stair case phenomena for homogeneous and also for the site dependent SQD system. We have predicted interesting physics for different stair case. We also conclude that the appearance of Cooper pair stair case is not only related with the Coulomb blocked phenomena but also related with the Josephson junction tunneling.

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